

Are rogue waves really unexpected?

FRANCESCO FEDELE*

School of Civil and Environmental Engineering, School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA, USA.

ABSTRACT

An unexpected wave is defined by Gemmrich and Garrett (2008) as a wave that is much taller than a set of neighboring waves. Their definition of "unexpected" refers to a wave that is not anticipated by a casual observer. Clearly, unexpected waves defined in this way are predictable in a statistical sense. They can occur relatively often with a small or moderate crest height, but large unexpected waves that are rogue are rare. Here, this concept is elaborated and statistically described based on a third-order nonlinear model. In particular, the conditional return period of an unexpected wave whose crest exceeds a given threshold is developed. This definition leads to greater return periods or on average less frequent occurrences of unexpected waves than those implied by the conventional return periods not conditioned on a reference threshold. Ultimately, it appears that a rogue wave that is also unexpected would have a lower occurrence frequency than that of a usual rogue wave. As specific applications, the Andrea and WACSIS rogue wave events are examined in detail. Both waves appeared without warning and their crests were nearly 2-times larger than the surrounding $O(10)$ wave crests, and thus unexpected. The two crest heights are nearly the same as the threshold $h_{0.3 \cdot 10^6} \sim 1.6H_s$ exceeded on average once every $0.3 \cdot 10^6$ waves, where H_s is the significant wave height. In contrast, the Andrea and WACSIS events, as both rogue and unexpected, would occur slightly less often and on average once every $3 \cdot 10^6$ and $0.6 \cdot 10^6$ waves respectively.

1. Introduction

A rogue wave is defined as such if the crest-to-trough height is at least 2.2 times the significant wave height H_s or if the crest height exceeds the threshold $1.25H_s$, where $H_s = 4\sigma$ and σ is the standard deviation of surface elevations (Dysthe et al. 2008). Evidences given for the occurrence of such waves in nature include the Draupner and Andrea events. In particular, the Andrea wave was measured on November 9 2007 by a LASAR system mounted on the Ekofisk platform in the North Sea in a water depth of $d = 74$ m (Magnusson and Donelan 2013). The Draupner freak wave was measured by Statoil at a nearby platform in January 1995 (Haver 2001). In the last decade, the properties of the Draupner and Andrea waves have been extensively studied (Dysthe et al. (2008); Osborne (1995); Magnusson and Donelan (2013); Bitner-Gregersen et al. (2014); Dias et al. (2015) and references therein).

The Andrea wave occurred during a sea state with significant wave height $H_s = 4\sigma = 9.2$ m, mean period $T_0 = 13.2$ s and wavelength $L_0 = 220$ m. The Andrea crest height is $h = 1.63H_s = 15$ m and the crest-to-trough height $H = 2.3H_s = 21.1$ m. The sea state during which

the Draupner wave occurred had a significant wave height $H_s = 11.9$ m, mean period $T_0 = 13.1$ s and wavelength $L_0 = 250$ m. The Draupner crest height is $h = 18.5$ m ($h/H_s = 1.55$) and the associated crest-to-trough height $H = 25.6$ m ($H/H_s = 2.15$) (Magnusson and Donelan (2013)). Observations of such large extreme waves show that they tend to extend above the surrounding smaller waves without warning and thus unexpectedly. Further, both waves were twice as high as the immediately preceding as well as following groups of waves. In describing the unexpectedness of ocean waves, Gemmrich and Garrett (2008) define as unexpected a wave α -times larger than a set of one-sided (preceding) waves or two-sided (preceding and following) waves (see Fig. 1). Note that their definition of unexpectedness refers to the time interval of apparent calm before or during which a wave is much taller than the neighboring waves. Hereafter, the term "unexpected" refers to a wave that is not anticipated by a casual observer as emphasized by Gemmrich and Garrett (2010). Clearly, unexpected waves defined in this way are predictable in a statistical sense as one can estimate the associated return period or frequency of occurrence.

Indeed, unexpected waves occur often with a small or average wave height, but they are rarely the largest waves in a record or rogue waves (Gemmrich and Garrett 2010). In this regard, Gemmrich and Garrett (2008) performed

*Corresponding author address: Georgia Institute of Technology
Atlanta, GA 30332, USA.
E-mail: fedele@gatech.edu

Monte Carlo simulations of second order nonlinear seas characterized with the typical JONSWAP ocean spectrum and initial homogeneous random conditions. They estimated that a wave with height at least twice that of any of the preceding 30 waves occurs once every 10^5 waves on average. Also unexpected crest heights are more probable than unexpected wave heights as they occur on average once every $7 \cdot 10^4$ in Gaussian seas and once every 10^4 waves in second-order nonlinear seas (see Fig. 2 in Gemmrich and Garrett (2008)). Thus, their numerical predictions indicate that in weakly nonlinear seas unexpected waves occur frequently and more often than in Gaussian seas.

Further, Gemmrich and Garrett (2008) noted in their simulations that among the unexpected waves 2-times larger than the surrounding 30 waves, only about $q = 10 - 20\%$ were rogues. With reference to second order crest heights, this means that in a sample population of 10^6 waves a set of 100 waves are unexpected, as they occur once every 10^4 waves on average. However, only about 10 – 20 waves of the set have crest heights that are rogue, i.e. larger than $1.34H_s$ as the rogue threshold adopted by Gemmrich and Garrett (2008). This implies that unexpected wave crests that are rogue would occur less often, i.e. once every 10^5 waves on average. Further, the percentage q of rogue occurrences can be interpreted as the probability that the crest of an unexpected wave exceeds the threshold $1.34H_s$. Consequently, unexpected crest heights larger than $1.34H_s$ would occur rarely.

The preceding results provide the principal motivation here to consider a statistical model for describing unexpected waves and their rogueness. We will show that Gemmrich and Garrett (2008)'s definition of return period is unconditional. In particular, it is the harmonic mean of the return periods of all unexpected waves with any amplitude. Thus, unexpected waves of moderate amplitude occur relatively often. However, unexpected waves that are rogue have a lower occurrence frequency, and this is in agreement with Gemmrich and Garrett (2008)'s numerical predictions.

The remainder of the paper is structured as follows. First, we introduce a new theoretical model for the statistics of unexpected waves that accounts for both second- and third-order nonlinearities. We also study the effects of nonstationarity and stochastic dependence among successive waves. In particular, we present analytical solutions for the return period of unexpected waves and associated unconditional and conditional averages for crest and wave heights. Then, the conceptual framework is validated by way of Monte Carlo simulations and the theoretical predictions are compared to oceanic measurements. As a specific application here, we capitalize on the numerical simulations of the Andrea sea state (Bitner-Gregersen et al. 2014; Dias et al. 2015) and examine the unexpectedness of the Andrea wave in detail. Summary and conclusions follow subsequently.

2. Statistics of unexpected waves

Consider the exceedance probability distribution of wave crests characterized by third-order nonlinearities and described by (Tayfun and Fedele 2007)

$$P(x) = \Pr[h > xH_s] = \exp(-8x_0^2) [1 + \Lambda x_0^2 (4x_0^2 - 1)], \quad (1)$$

where $x = h/H_s$ is the crest amplitude h scaled by the significant wave height $H_s = 4\sigma$ and x_0 follows from the quadratic equation (Tayfun 1980)

$$x = x_0 + 2\mu x_0^2. \quad (2)$$

Here, the wave steepness $\mu = \lambda_3/3$ relates to the skewness of surface elevations (Fedele and Tayfun 2009) and the parameter

$$\Lambda = \lambda_{40} + 2\lambda_{22} + \lambda_{04} \quad (3)$$

is a measure of third-order nonlinearities as a function of the fourth order cumulants λ_{nm} of the wave surface η and its Hilbert transform $\hat{\eta}$ (Tayfun and Fedele 2007). Mori and Janssen (2006) assume the following relations between cumulants

$$\lambda_{22} = \lambda_{40}/3, \quad \lambda_{04} = \lambda_{40}, \quad (4)$$

which, to date, have been proven to hold for second-order narrowband waves only (Tayfun and Lo 1990). Then, Λ in Eq. (3) is approximated in terms of the excess kurtosis λ_{40} by

$$\Lambda_{\text{appr}} = \frac{8\lambda_{40}}{3}, \quad (5)$$

which will be used in this work. Then, Eq. (1) reduces to a modified Edgeworth-Rayleigh (MER) distribution (Mori and Janssen 2006). For realistic oceanic seas the kurtosis λ_{40} is mainly affected by bound nonlinearities (Annenkov and Shrira 2014; Fedele 2015b,a).

Consider now a time interval \mathcal{T} during which a stationary sequence of $N_w = \mathcal{T}/T_m$ consecutive waves occur on average. We assume that neighboring waves are stochastically independent. This assumption is convenient for the theoretic development of a probabilistic model. Furthermore, Borgman (1970) argues that "... *It seems reasonable to assume that a wave height is at most interdependent with the first several wave heights occurring before and after it and essentially independent with waves further back into the past or forward into the future*". We will show later that this is justified as long as the sea state is broadband so that the covariance function decays sufficiently rapid to zero after few wave periods and successive wave peaks decorrelate faster. Thus, in a sample of $N_a + 1$ successive waves it is irrelevant what wave is the unexpected wave larger than the surrounding waves. Indeed, any wave in the sample could be " p -sided" unexpected, i.e. α -times larger than the previous m waves and following $N_a - m$ waves, with $m = 1, \dots, N_a/2$ and $p = N_a/m$. For instance,

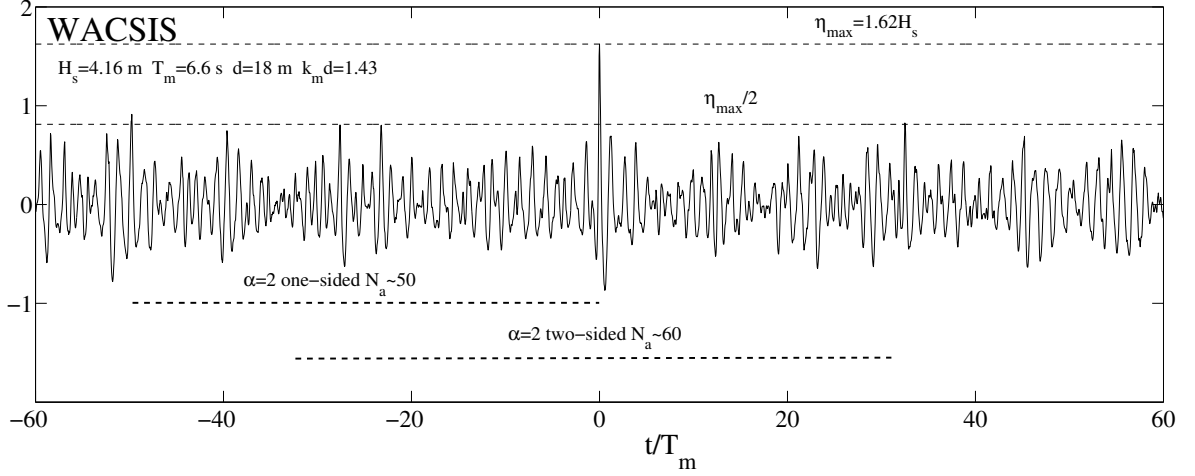


FIG. 1. WACSIS measurements: the observed largest crest height is $\alpha = 2$ -times larger than the crests of the one-sided (two-sided) $N_a \sim 50$ (60) waves. Wave parameters $H_s = 4.16$ m, $T_m = 6.6$ s, depth $d = 18$ m (Forristall et al. 2004).

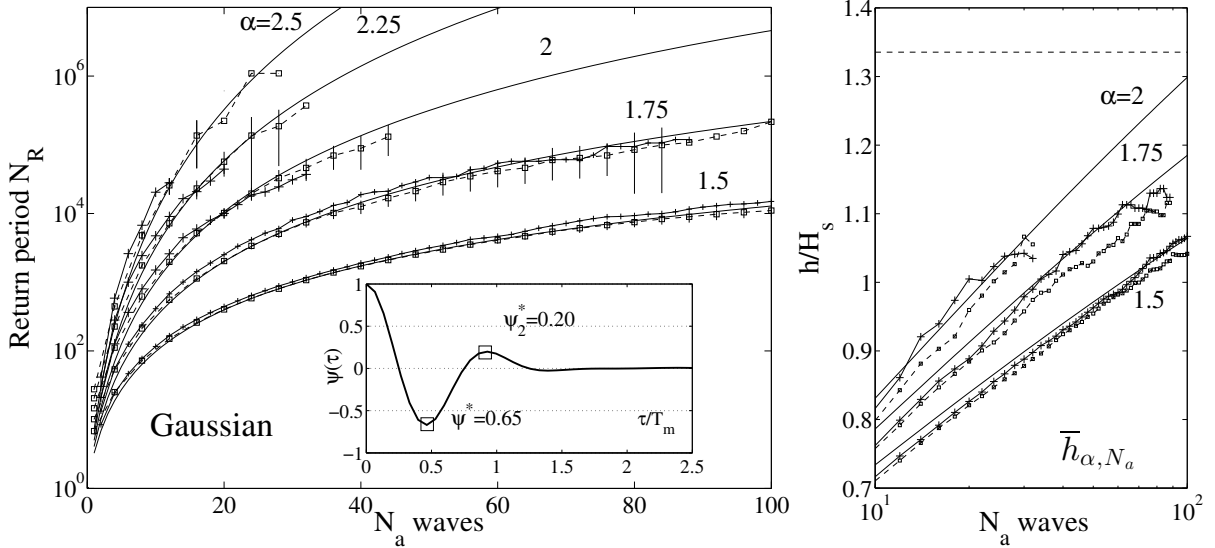


FIG. 2. Unexpected crest heights in broadband Gaussian seas. Left panel: empirical one-sided (thin dashed line with \square) and two-sided (thin solid line with $+$, N_a even) unexpected wave statistics versus (solid line) predicted theoretical unconditional return period N_R in number of waves of a wave whose crest height is α -times larger than the surrounding N_a waves for increasing values of $\alpha = 1.5, 2$ and 2.5 . Confidence bands are also shown. Right panel: empirical one-sided (thin dashed line with \square) and two-sided (thin solid line with $+$, N_a even) unexpected wave statistics versus theoretical predictions (solid line) of the mean crest height \bar{h}_{α, N_a} of a wave whose crest height is α -times larger than surrounding N_a waves for $\alpha = 1.5, 1.75$, and 2 . Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor $\gamma = 1$), mean period $T_m = 8.3$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameters $\psi^* = 0.65$, $\psi_2^* = 0.20$ and simulated $\sim 10^6$ waves (see left panel inset). The theoretical predictions accounting for the stochastic independence and dependence of successive crest heights are practically the same as the sea state is broadband.

the last wave in the sample could be larger than the preceding (one-sided) N_a waves ($m = N_a$ and $p = 1$), or the central wave could extend above the preceding and following (two-sided) $m = N_a/2$ waves ($p = 2$ and N_a even) (see Fig. 1). Note that our definition of two-sided unexpected-

ness is different than that in Gemmrich and Garrett (2008) as they consider N_a waves on each side.

Clearly, the statistics of one- and two-sided unexpected waves, or more generally the p -sided statistics are the same if stochastic independence of successive waves holds. On this basis, the fraction of waves $n(x; \alpha, N_a)$ that

have a dimensionless crest height h/H_s within the interval $(x, x+dx)$ and that is α -times larger than any of the surrounding N_a waves is given by

$$n(x; \alpha, N_a)dx = \left[1 - P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx, \quad (6)$$

where $P(x)$ is the exceedance probability given in Eq. (1) and

$$p(x) = -\frac{dP}{dx} \quad (7)$$

is the pdf of x . Then the probability that the crest height ξ is in $(x, x+dx)$ follows as

$$p_h(x; \alpha, N_a)dx = \frac{n(x; \alpha, N_a)dx}{n(\alpha, N_a)}, \quad (8)$$

where $n(\alpha, N_a)$ is the fraction of waves whose crest height is α -times larger than the surrounding N_a waves, namely

$$n(\alpha, N_a) = \int_0^\infty n(x; \alpha, N_a)dx = \int_0^\infty \left[1 - P\left(\frac{x}{\alpha}\right)\right]^{N_a} p(x)dx. \quad (9)$$

By definition, the unconditional return period R or the average time interval between two consecutive occurrences of the unexpected wave event \mathcal{E} is

$$R(\alpha, N_a) = \frac{\tau}{N_w n(\alpha, N_a)} = \frac{N_w T_m}{N_w n(\alpha, N_a)} = \frac{T_m}{n(\alpha, N_a)}. \quad (10)$$

Since T_m is the mean wave period, \mathcal{E} occurs on average once every N_R waves where

$$N_R(\alpha, N_a) = \frac{1}{n(\alpha, N_a)}. \quad (11)$$

Another statistical interpretation of the unconditional return period N_R is as follows. Consider the average number of unexpected waves $n_j(\alpha, N_a)\Delta x$ with a crest height between $x_j - \Delta x/2$ and $x_j + \Delta x/2$, where $\Delta x \ll 1$ is small and x_j are increasing amplitudes starting from $x_1 = 0$, i.e. $x_{j+1} > x_j$, for $j = 1, \dots$. Then,

$$N_{R,j}(\alpha, N_a) = \frac{1}{n_j(\alpha, N_a)\Delta x}$$

is the return period of an unexpected wave whose crest height is nearly x_j . Then, Eq. (11) is approximated as

$$N_R(\alpha, N_a) \simeq \frac{1}{\sum_{j=1}^\infty n_j(\alpha, N_a)\Delta x} = \frac{1}{\sum_{j=1}^\infty \frac{1}{N_{R,j}(\alpha, N_a)}}, \quad (12)$$

which reveals that N_R is the harmonic mean of the return periods $N_{R,j}$ of all unexpected waves with any crest height.

The associated mean crest height of a wave α -times larger than the surrounding N_a waves follows from Eq. (8) as

$$\bar{h}_{\alpha, N_a} = H_s \int_0^\infty x p_h(x; \alpha, N_a)dx. \quad (13)$$

For comparison purposes, we also consider the standard statistics $\bar{h}_{\max, n}$, h_n and $h_{1/n}$ for crest heights (Tayfun and Fedele (2007)). In particular, $\bar{h}_{\max, n}$ is the mean maximum crest height of a sample of n waves

$$\bar{h}_{\max, n} = H_s \int_0^\infty \{1 - [1 - P(x)]^n\} dx, \quad (14)$$

which admits Gumbel-type asymptotic approximations (Tayfun and Fedele (2007); Fedele (2015a)). Further, h_n is the threshold exceeded by the $1/n$ fraction of largest crest heights and it satisfies

$$P(h_n/H_s) = \frac{1}{n}, \quad (15)$$

where $P(x)$ is the unconditional nonlinear probability of exceedance for crest heights given in Eq. (1). The statistics $h_{1/n}$ is the conditional mean $\bar{h} | h > h_{1/n}$, namely the average of the $1/n$ fraction of largest crest heights

$$h_{1/n} = h_n + nH_s \int_{h_n}^\infty P(x)dx. \quad (16)$$

One can show that $\bar{h}_{\max, n}$ is always smaller than $h_{1/n}$ and they tend to be the same as n increases (Tayfun and Fedele 2007).

We also consider the standard conditional return period $N_h(\xi)$ (in number of waves) of a wave whose crest exceeds the threshold $h = \xi H_s$, namely

$$N_h(\xi) = \frac{1}{\Pr[h > \xi H_s]} = \frac{1}{P(\xi)}, \quad (17)$$

where the exceedance probability $P(\xi)$ is that in Eq.(1). From Eq. (15), the threshold h_n exceeded with probability $1/n$ implies that $N_h(h_n/H_s) = n$, i.e. on average h_n is exceeded once every n waves.

Similar statistics for the crest-to-trough height $y = H/H_s$ of unexpected waves follow by replacing the crest exceedance probability P in Eq. (1) with the generalized Boccotti distribution (Alkhalidi and Tayfun 2013)

$$P_H(y) = \Pr[H > yH_s] = c_0 \exp\left(-\frac{4y^2}{1+\psi^*}\right) \left[1 + \frac{\Lambda y^2}{1+\psi^*} \left(\frac{y^2}{1+\psi^*} - \frac{1}{2}\right)\right], \quad (18)$$

where

$$c_0 = \frac{1 + \ddot{\psi}^*}{\sqrt{2\ddot{\psi}^*(1+\psi^*)}},$$

and $\psi^* = \psi(\tau^*)$ is the absolute value of the first minimum of the normalized covariance function $\psi(\tau) = \frac{\eta(t)\eta(t+\tau)}{\sigma^2}$ of the zero-mean random wave process $\eta(t)$, which is attained at $\tau = \tau^*$ and $\ddot{\psi}^*$ the corresponding second derivative (Boccotti 2000).

The corresponding linear statistics of unexpected wave crests follow by setting $\mu = 0$ and $\Lambda = 0$ in Eq. (1), or

$\Lambda = 0$ in Eq. (18) for wave heights. These will hereafter be differentiated with the superscript L . In the following, we will not dwell that much on unexpected wave heights, but our main focus will be the statistics of unexpected crests in typical oceanic sea states.

Finally, we point out that our present theory of unexpected waves can be generalized to space-time extremes drawing on Fedele (2012), but this is beyond the scope of this paper.

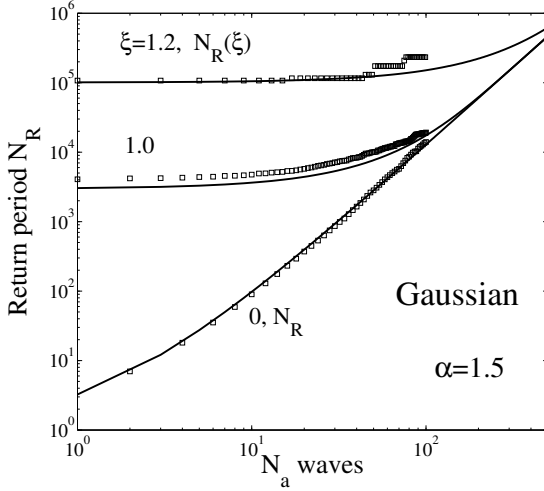


FIG. 3. Conditional return period of large unexpected waves in Gaussian seas: (square) empirical one-sided unexpected wave statistics versus (solid lines) predicted theoretical conditional return periods $N_R(\xi)$ in number of waves of unexpected waves whose crest height is greater than ξH_s and $\alpha = 1.5$ -times larger than the surrounding N_a waves for $\xi = 0, 1.0$ and 1.2 . Note that $N_R(\xi = 0)$ is the unconditional return period N_R . Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor $\gamma = 1$), mean period $T_m = 8.3$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameters $\psi^* = 0.65$, $\psi_2^* = 0.20$ and simulated $\sim 10^6$ waves. The predictions accounting for the stochastic independence and dependence of successive crest heights are practically the same as the sea state is broadbanded.

a. Stochastic dependence of successive waves

The statistics of unexpected waves presented so far does not take into account the stochastic dependence of neighboring waves or wave groupness. Clearly, for large crest heights, as argued by Borgman (1970), one expects that only few neighboring crests are more or less correlated (Watson 1954). To quantify this, we draw on Fedele (2005) and model a stationary sequence of wave crests $\{x_j = \frac{h_j}{H_s}\}_{j=1, N_w}$ as a one-step memory Markov chain, where each crest height x_j is only stochastically dependent on the preceding crest height x_{j-1} , that is

$$p(x_j | x_{j-1}, x_{j-2}, \dots, x_2, x_1) = p(x_j | x_{j-1}).$$

Since the sequence is stationary, the conditional pdf $p(x_j | x_{j-1})$ is the same for any j , say $p(x_2 | x_1) = p(x_1, x_2) / p(x_1)$, where the crest x_1 precedes x_2 , and $p(x_1, x_2)$ and $p(x_1)$ are the associated joint and marginal pdfs.

On these assumptions, following Fedele (2005) the fraction of waves $n(x; \alpha, N_a)$ that have a dimensionless crest height within the interval $(x, x + dx)$ and that is α -times larger than any of the surrounding N_a waves is given by

$$n(x; \alpha, N_a) dx = \left[\Pr \left(x_2 < \frac{x}{\alpha} \mid x_1 < \frac{x}{\alpha} \right) \right]^{N_a-1} \cdot \Pr \left(x_2 < \frac{x}{\alpha} \mid x_1 = x \right) p(x) dx, \quad (19)$$

where

$$\Pr \left(x_2 < \frac{x}{\alpha} \mid x_1 < \frac{x}{\alpha} \right) = \frac{\int_0^{\frac{x}{\alpha}} \int_0^{\frac{x}{\alpha}} p(x_1, x_2) dx_1 dx_2}{\int_0^{\frac{x}{\alpha}} p(x_1) dx_1},$$

and

$$\Pr \left(x_2 < \frac{x}{\alpha} \mid x_1 = x \right) = \frac{\int_0^{\frac{x}{\alpha}} p(x, x_2) dx_2}{p(x)}.$$

Then, the return period $R(\alpha, N_a)$ of unexpected wave crests follows from Eqs. (9) and (11). Clearly, if successive waves were stochastically independent, $p(x_1, x_2) = p(x_1)p(x_2)$ and Eq. (19) reduces to Eq. (6) for the stationary case.

The theoretical probability structure of two consecutive wave crests is known for Gaussian processes and it is given by the bivariate Rayleigh distribution (Fedele 2005)

$$p_R(x_1, x_2) = 256 \frac{x_1 x_2}{1 - k^2} \exp \left[-8 \frac{x_1^2 + x_2^2}{1 - k^2} \right] I_0 \left(16k \frac{x_1 x_2}{1 - k^2} \right), \quad (20)$$

where $I_0(y)$ is the modified Bessel function and the parameter $k = \psi(\tau_2^*) = \psi_2^*$ with τ_2^* the abscissa of the second absolute maximum of the normalized covariance function $\psi(\tau)$ of the zero-mean random wave process (Fedele 2005). Further, the marginal pdf

$$p_R(x_1) = \int_0^\infty p_W(x_1, x_2) dx_2 = 16x_1 \exp(-8x_1^2)$$

is the univariate Rayleigh distribution. As ψ_2^* tends to zero, successive crests become stochastically independent and the sea state tends to be broadbanded. Thus, we expect that stochastic dependence of waves is dominant in very narrowband sea states, where $\psi_2^* \rightarrow 1$. In particular, our numerical simulations discussed later in section 4 suggest that the dependence of consecutive crests in Gaussian seas is dominant when $\psi_2^* > 0.7$. This condition corresponds to unrealistic oceanic sea states characterized by a Jonswap spectrum with a peak enhancement factor $\gamma > 100$ and very narrowbanded as the spectral bandwidth $\nu < 0.1$. For typical oceanic seas, $\nu \sim 0.3 - 0.5$ and $\psi_2^* \sim 0.2 - 0.5$,

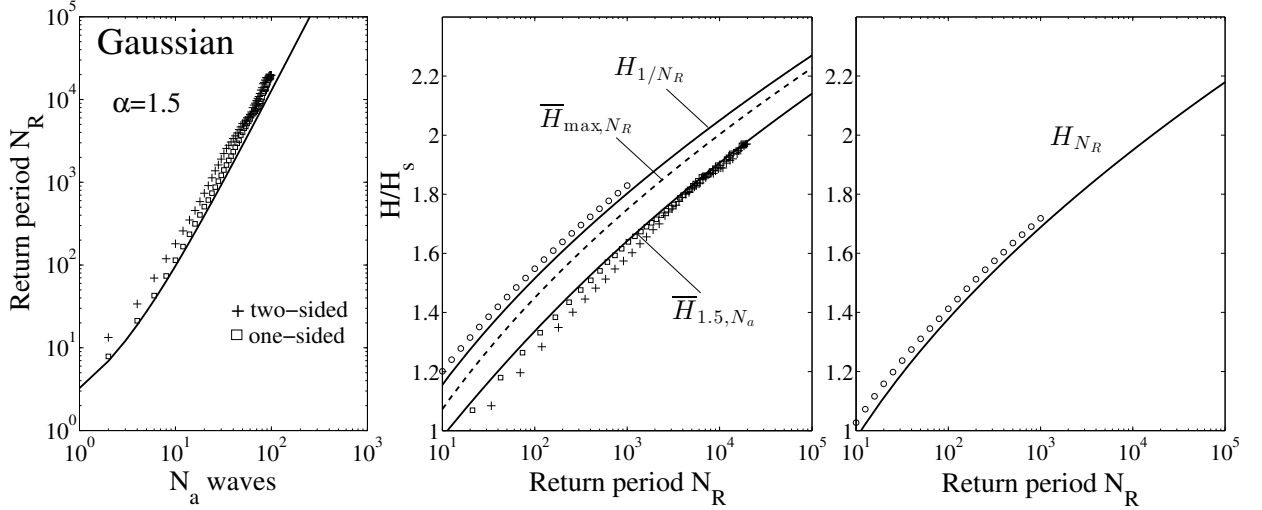


FIG. 4. Unexpected wave heights in Gaussian seas: (Left panel) predicted theoretical unconditional return period N_R in number of waves (solid line) versus empirical one-sided (+) and two-sided (\square , N_a even) statistics as a function of the number N_a of surrounding waves for $\alpha = 1.5$; (center panel) predicted mean unexpected wave height $\bar{H}_{1.5,N_R}$ versus observations as a function of the return period N_R . For comparison purposes, predicted mean wave height \bar{H}_{\max,N_R} , conditional mean H_{1/N_R} and (right panel) threshold H_{N_R} versus observations (circles) are also shown. Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor $\gamma = 1$), mean period $T_m = 8.3$ s, spectral bandwidth $\nu = 0.35$, Boccotti parameter $\psi^* = 0.65$ and simulated $\sim 10^6$ waves.

and successive waves can be assumed as stochastically independent.

Note that, the joint pdf of consecutive Gaussian wave crests in Eq. (20) can be generalized to account for second-order bound nonlinearities following Fedele and Tayfun (2009), but this is beyond the scope of this work. Since second-order bound harmonics are phase-locked to the Fourier components of the linear free surface, we expect the classical Tayfun's (1980) enhancement of successive linear crest amplitudes, but their dependence should be unaffected by second-order nonlinearities.

b. Nonstationarity

The statistics of unexpected waves formulated so far is valid for stationary sea states. In nonstationary seas, as those during storms, our present theory can be formalized as follows. From Eq. (8), the pdf of an unexpected wave crest height h generalizes to

$$p_h(x; \alpha, N_a)_{NS} = \int \cdots \int p_h(x; \alpha, N_a | b_1, \dots, b_M) \cdot p(b_1, \dots, b_M) db_1 \dots db_M, \quad (21)$$

where $\{b_j\}_{j=1,M}$ are M time-varying wave parameters, e.g. $\sigma, \mu, \lambda_{40}, \lambda_{22}$ and λ_{04} , the conditional pdf $p_h(x; \alpha, N_a | b_1, \dots, b_M)$ is the stationary pdf in Eq. (8) for given values of b_j and $p(b_1, \dots, b_M)$ is the joint pdf of the parameters, which encodes their time variability. Eq. (21) can be interpreted as the average value of

$p_h(x; \alpha, N_a | b_1, \dots, b_M)$ with respect to the random variables b_j , that is

$$p_h(x; \alpha, N_a)_{NS} = \overline{p_h(x; \alpha, N_a | \mathbf{b})}^{\mathbf{b}},$$

where the vector $\mathbf{b} = [b_1, \dots, b_M]$ and the labeled overbar denotes statistical average with respect to \mathbf{b} only. Taylor-expanding around the mean $\bar{\mathbf{b}} = [\bar{b}_1, \dots, \bar{b}_M]$, up to second order, yields

$$p_h(x; \alpha, N_a)_{NS} \simeq \overline{p_h(x; \alpha, N_a | \bar{\mathbf{b}})}^{\mathbf{b}} + \overline{\sum_j \mathbf{g}^T (\mathbf{b} - \bar{\mathbf{b}})}^{\mathbf{b}} + \frac{\overline{(\mathbf{b} - \bar{\mathbf{b}})^T \mathbf{H}(\bar{\mathbf{b}}) (\mathbf{b} - \bar{\mathbf{b}})}}{2}, \quad (22)$$

where the superscript T denotes matrix transposition, the vector \mathbf{g} has entries

$$[\mathbf{g}]_j = \left. \frac{\partial p_h(x; \alpha, N_a | b_1, \dots, b_M)}{\partial b_j} \right|_{\mathbf{b}=\bar{\mathbf{b}}}$$

and the Hessian matrix

$$[\mathbf{H}(\bar{\mathbf{b}})]_{rs} = H_{rs} = \left. \frac{\partial^2 p_h(x; \alpha, N_a | b_1, \dots, b_M)}{\partial b_r \partial b_s} \right|_{\mathbf{b}=\bar{\mathbf{b}}}.$$

Taking the averages in Eq. (22) yields

$$p_h(x; \alpha, N_a)_{NS} \simeq p_h(x; \alpha, N_a | \bar{b}_1, \dots, \bar{b}_M) + \sum_{rs} \left[\left. \frac{\partial^2 p_h(x; \alpha, N_a | b_1, \dots, b_M)}{\partial b_r \partial b_s} \right|_{\mathbf{b}=\bar{\mathbf{b}}} \right] B_{rs}, \quad (23)$$

where

$$B_{rs} = \overline{(b_r - \bar{b}_r)(b_s - \bar{b}_s)}^\beta$$

are correlation coefficients, in particular $B_{rr} = \sigma_{b_r}^2$ is the variance of b_r . These can be easily estimated from the nonstationary time series. Thus, p_h is the sum of i) the pdf in Eq. (8) evaluated using the mean parameters $\bar{\mathbf{b}}$ and ii) additional terms that account for the spreading of the parameters from their mean. A similar formula can be obtained for $n(\alpha, N_a)$ in Eq. (6). The statistical moments of p_h can then be obtained by integrating Eq. (23) and the nonstationary return period N_R^{NS} follows from Eq. (11).

In our applications (see section 4), time wave measurements at a point are subdivided in a sequence of optimal 30-min intervals during which the sea state can be assumed as stationary. We observed that shorter time intervals lead to unstable estimates of higher order moments, whereas longer intervals violate the stationarity assumption. The variability of the standard deviation σ was taken into account by normalizing the surface height measurements in each 30-min interval by the respective observed σ . In our data analysis, wave parameters are estimated as the average values over the available time record. Then, the statistics of unexpected waves can be based on Eq. (23), where the B_{rs} terms accounting for non-stationarity are neglected.

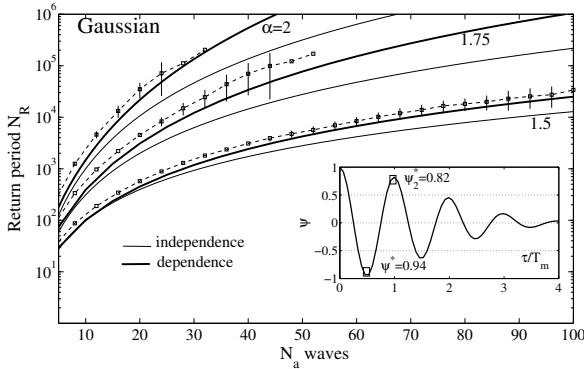


FIG. 5. The role of stochastic wave dependence to the unexpectedness of crest heights in narrowband Gaussian seas: (thin dashed line with \square) empirical one-sided unexpected wave statistics versus (solid lines) predicted theoretical unconditional return periods N_R in number of waves for (thin line) independent and (thick line) dependent crest heights of a wave whose crest height is α -times larger than the surrounding N_a waves; $\alpha = 1.5, 1.75$ and 2 . Confidence bands are also shown. Sea state parameters: Gaussian spectrum with spectral bandwidth $\nu = 0.1$ (similar to a Jonswap spectrum with peak enhancement factor $\gamma \sim 300$), mean period $T_m = 8.3$ s, Boccotti parameters $\psi^* = 0.94$, $\psi_2^* = 0.82$ and simulated $\sim 10^6$ waves.

3. Are rogue waves really unexpected?

Our interest is to describe statistically the occurrence of rogue waves with crest heights larger than $1.25H_s$ (Dysthe

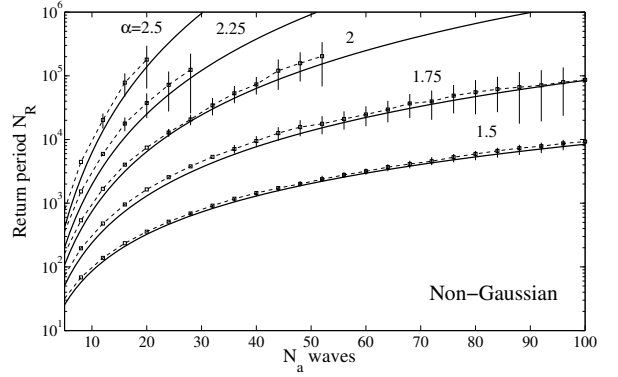


FIG. 6. Unexpected crest heights in unidirectional second-order random seas. Empirical one-sided (thin dashed lines with \square) unexpected wave statistics versus (thick solid lines) predicted theoretical unconditional return period N_R in number of waves of a wave whose crest height is α -times larger than the surrounding N_a waves for increasing values of $\alpha = 1.5, 1.75, 2, 2.25$ and 2.5 . Confidence bands are also shown. Sea state parameters: fully developed JONSWAP spectrum (peak enhancement factor $\gamma = 1$), mean period $T_m = 8.3$ s, spectral bandwidth $\nu = 0.35$, Tayfun steepness $\mu_m = 0.06$ and simulated $\sim 10^6$ waves. The theoretical predictions accounting for the stochastic independence and dependence of successive crest heights are practically the same as the sea state is broadbanded.

et al. 2008). For example, observations indicate that the Andrea rogue wave appeared without warning suddenly, attained a crest height $h_{obs} = 1.62H_s$, and it was as nearly two-times larger than the surrounding $O(30)$ waves (Magnusson and Donelan 2013). Thus, the Andrea wave is unexpected in accordance with the definition of Gemmrich and Garrett (2008). However, as it will be discussed later in section 6, an application of our present theory using Eq. (11) predicts that a wave with a crest height at least twice as that of any of the surrounding $N_a = 30$ waves occurs on average once every $N_R \sim 10^4$ waves. This is clearly observed in the left panel of Fig. 10. Further, the right panel of the same Figure shows that the actual Andrea crest height is nearly the same as the threshold $h_{0.3 \cdot 10^6} \sim 1.6H_s$ exceeded by the $1/(0.3 \cdot 10^6)$ fraction of largest crests. Eq. (17) also suggests that the Andrea wave is likely a rare event as the crest threshold $1.6H_s$ is exceeded once every $N_h = 0.3 \cdot 10^6$ waves on average. In contrast, our present theory predicts that the Andrea event would occur relatively often as an unexpected wave, i.e. on average once every $N_R \sim 10^4$ waves.

The difference in occurrence rates is explained by first noting that the return period N_R is the average time interval between two consecutive waves whose crest height h , of any possible amplitude, is α -times larger than the surrounding N_a wave crests. In other words, Eq. (12) reveals that N_R is the harmonic mean of the return periods of all unexpected waves of any crest amplitude, and it is smaller than the return period of large (rare) unexpected waves. Thus, unexpected waves as defined by Gemm-

rich and Garrett (2008) occur relatively often with small or moderate amplitude. However, unexpected waves that are rogue are rare, in agreement with their numerical predictions (see also Gemmrich and Garrett (2010)).

To quantify the difference in occurrence frequencies of small and large unexpected waves, it is natural to define the conditional return period $N_R(\xi, \alpha, N_a)$ of an unexpected wave whose crest height h exceeds the threshold ξH_s and it is α -times larger than the surrounding N_a wave crests. This is given by

$$N_R(\xi; \alpha, N_a) = \frac{1}{\int_{\xi}^{\infty} n(x; \alpha, N_a) dx} = \frac{N_R(\alpha, N_a)}{P_h(\xi; \alpha, N_a)}, \quad (24)$$

where

$$P_h(x; \alpha, N_a) = \int_x^{\infty} p_h(s; \alpha, N_a) ds \quad (25)$$

is the exceedance probability of the unexpected crest height h from Eq. (8). Clearly, for given α and N_a the conditional return period $N_R(\xi)$ is always greater than the unconditional N_R for any $\xi > 0$, and they are the same if $\xi = 0$. The left panel of Fig. 10 shows that the Andrea rogue wave as an unexpected wave that exceeds $\xi H_s = 1.6 H_s$ would occur rarely, i.e. on average once every $N_R(\xi = 1.6) \sim 6 \cdot 10^6$. Instead, unexpected waves of any amplitude occur more often, and on average once every $N_R \sim 10^4$.

Clearly, the Andrea wave is both rogue and unexpected, i.e. its crest is larger than the crests of surrounding waves and it exceeds the threshold $1.25 H_s$ (Dysthe et al. 2008). What is the occurrence frequency of such a bivariate event in comparison to being only rogue as an univariate event?

From Eq. (24) the following inequality holds

$$N_R(\xi) \geq \frac{1}{\int_{\xi}^{\infty} p(x) dx} = \frac{1}{P(\xi)} = N_h(\xi), \quad (26)$$

where we have used $n(x; \alpha, N_a) \leq p(x)$ from Eq. (6). Here, $N_h(\xi)$ is defined in Eq. (17)) as the standard unconditional return period (in number of waves) of a wave whose crest exceeds the threshold $h = \xi H_s$. Thus, a wave whose crest is both larger than ξH_s and unexpected (as being larger than the surrounding waves) has a lower occurrence frequency than a wave whose crest is just larger than the same threshold.

The preceding results imply that a rogue wave that is also unexpected has a lower occurrence frequency than just being rogue. For example, for the Andrea sea state the return period of a crest larger than $h_n = 1.6 H_s$ is $N_h(h_n) = 0.3 \cdot 10^6$. This is smaller than the return period N_R of an unexpected wave exceeding the same threshold, i.e. $N_R(\xi = 1.6) \sim 6 \cdot 10^6$ (see left panel of Fig. 10). Similar conclusions hold for the WACSYS rogue wave (see section 4).

4. Verification and comparisons

a. Monte Carlo simulations of Gaussian seas

Drawing on Gemmrich and Garrett (2008), we performed Monte Carlo simulations of a Gaussian sea described by the average JONSWAP spectrum with peak enhancement factor $\gamma = 1$. The sea state is broadbanded with mean period $T_m = 8.3$ s, peak period $T_p = 10$ s, spectral bandwidth $\nu \sim 0.35$ and Boccotti parameters $\psi^* = 0.65$, $\psi_2^* = 0.3$ (see covariance function in the panel inset of Fig. 2). A long time series of wave surface displacements was randomly generated containing a total of $\sim 10^6$ waves, from which unexpected waves were sampled. As the sea state is broadbanded, our theoretical predictions can be based on Eqs. (9) and (11) assuming the stochastic independence of successive crest heights.

The left panel of Fig. 2 shows the empirical return period $N_R = R/T_m$ in number of waves of both one-sided (thin dashed line) and two-sided (thin solid line) unexpected wave crests as a function of the surrounding N_a waves for different values of α (N_a even for the two-sided statistics). The two statistics are roughly the same with two-sided unexpected waves slightly less frequent than the one-sided waves. Note that for the two-sided unexpectedness Gemmrich and Garrett (2008) consider N_a waves on each side, thus their two-sided return period is larger than ours. Shown in the right panel of Fig. 2 are also the empirical statistics of mean crest heights in comparison to our theoretical predictions for stochastically independent waves. In particular, we note that the mean crest height of two-sided unexpected waves is slightly smaller than that of one-sided waves, especially as α increases. Further, in Fig. 3 there are shown the predicted conditional return periods $N_R(\xi)$ (solid lines) of an unexpected wave whose crest height is greater than ξH_s for $\xi = 0, 1.0$ and 1.2 ($\alpha = 1.5$). Note that $N_R(\xi = 0)$ is the unconditional return period N_R . We find a fair agreement with the empirical one-sided unexpected wave statistics (squares). For $\alpha = 2$ and $N_a = 30$ our predicted return period is $N_R \sim 6 \cdot 10^4$ and in fair agreement with the linear predictions ($\sim 7 \cdot 10^4$) by Gemmrich and Garrett (2008) as shown in their Fig. 2. As regard to unexpected crest-to-trough heights, our theoretical model fairly predicts the empirical wave height statistics from simulations as clearly seen in Fig. 4.

In the above comparisons, the fair agreement with our theoretical predictions indicates that the stochastic independence of waves holds approximately as the sea state is broadbanded. However, in very narrowband seas the stochastic dependence of neighboring waves cannot be neglected. Indeed, consider a linear sea state characterized with a Gaussian spectrum with spectral bandwidth $\nu = 0.1$. This is similar to an unrealistic Jonswap spectrum with peak enhancement factor $\gamma \sim 300$. From the panel inset of Fig. 5, the Boccotti parameters are $\psi^* = 0.94$ and

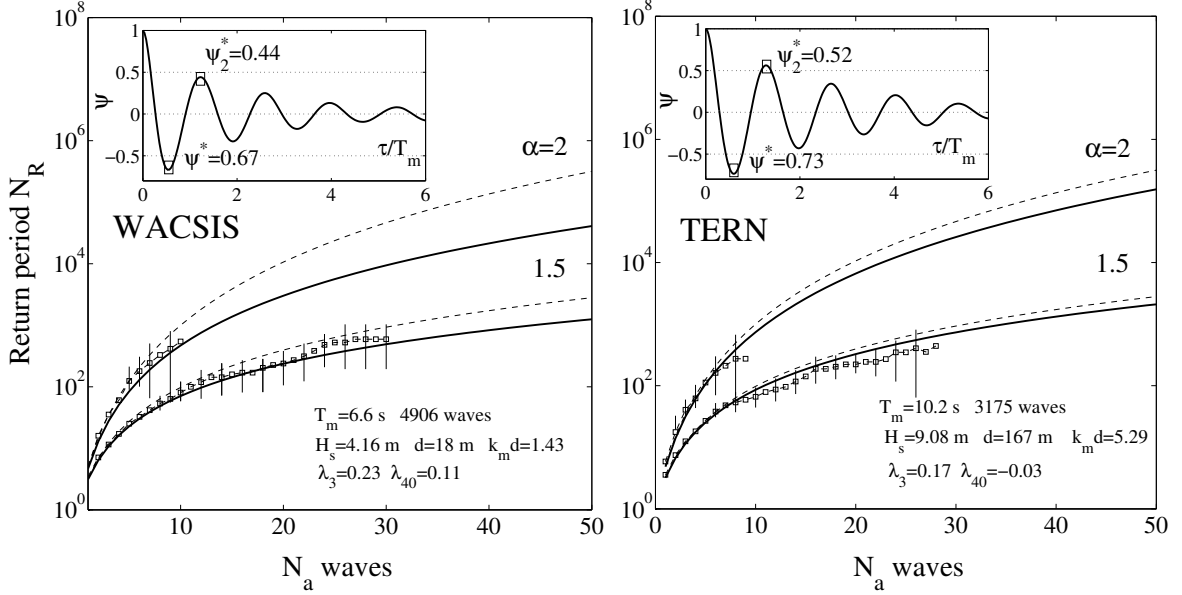


FIG. 7. Left panel: WACIS, predicted theoretical nonlinear unconditional return period N_R in number of waves (solid line) of a wave whose crest height is α -times larger than the surrounding N_a waves, linear predictions (dash lines) and empirical one-sided observed statistics (\square) for $\alpha = 1.5$ and 2. Confidence bands are also shown. Right panel: same for TERN measurements. Statistical parameters are taken from Tayfun (2006); Tayfun and Fedele (2007).

$\psi_2^* = 0.81$ indicating a strong correlation between consecutive waves. Indeed, from the same figure the empirical one-sided (square) unexpected wave statistics tends to agree with our predicted theoretical return period N_R for dependent waves (thick solid line) computed using Eqs. (19) and (11). Instead, our predictions for independent waves (thin solid line) are less conservative, where we use Eqs. (9) and (11).

b. Monte Carlo simulations of second-order random seas

Drawing on Tayfun and Fedele (2007), we performed Monte Carlo simulations of unidirectional second-order broadband random seas in deep water described by the same average JONSWAP spectrum introduced in the previous section for simulating Gaussian seas. The associated Tayfun (1980) steepness $\mu = \lambda_3/3 \sim 0.06$, where λ_3 is the skewness of surface elevations (Fedele and Tayfun 2009). Our theoretical predictions are based on Eqs. (9) and (11) and assume the stochastic independence of successive crest heights as the sea state is broadband.

In Fig. 6 it is shown the comparison between the empirical return period $N_R = R/T_m$ in number of waves of one-sided (squares) unexpected wave crests and theoretical predictions from our model as a function of the surrounding N_a waves for different values of α . For $\alpha = 2$ and $N_a = 30$ our predicted second-order return period $N_R \sim 2 \cdot 10^4$ is shorter than the linear counterpart ($\sim 6 \cdot 10^4$) for Gaussian seas (see Fig. 2) as nonlinearities enhance crest heights (Tayfun and Fedele 2007; Fedele and Tayfun

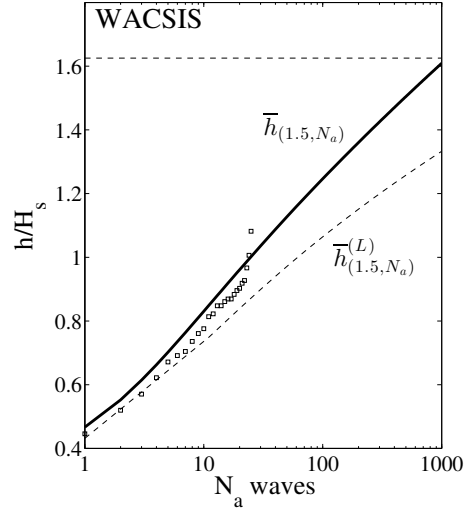


FIG. 8. WACIS unexpected wave crest heights: predicted theoretical nonlinear (solid line) and linear (dash line) mean heights \bar{h}_{α, N_a} and $\bar{h}_{\alpha, N_a}^{(L)}$ as a function of the number N_a of surrounding waves versus empirical one-sided statistics (squares) for $\alpha = 1.5$. Horizontal line denotes the observed maximum crest height $1.62H_s$. Wave parameters $H_s = 4.16$ m, $T_m = 6.6$ s, depth $d = 18$ m (Forristall et al. 2004). Average wave parameters are taken from Tayfun (2006); Tayfun and Fedele (2007), in particular skewness $\lambda_3 \sim 0.23$ and excess kurtosis $\lambda_{40} \sim 0.11$.

2009). Further, our second-order predictions fairly agree with those by Gemmrich and Garrett (2008) in their Fig. 2.

For example, they predict a slightly shorter nonlinear period $N_R \sim 10^4$ for $\alpha = 2$ and $N_a = 30$. This is because their second-order correction for crest heights is based on the narrowband assumption of the sea state. This yields a slightly overestimation of crest heights shortening N_R . In contrast, our simulated sea states are based on the exact second-order solution for unidirectional broadband waves in deep water (Tayfun 1980).

c. Oceanic observations

We will analyze two data sets. The first comprises 9 h of measurements gathered during a severe storm in January, 1993 with a Marex radar from the Tern platform located in the northern North Sea in a water depth of $d = 167$ m. We refer to Forristall (2000) for further details on the data set, hereafter referred to as TERN. The second data set is from the Wave Crest Sensor Intercomparison Study (WACSIS) (Forristall et al. (2004)). It consists of 5 h of measurements gathered in January, 1998 with a Baylor wave staff from Meetpost Noordwijk in the southern North Sea (average water depth $d = 18$ m). Tayfun (2006) and Tayfun and Fedele (2007) elaborated both data sets and provided accurate estimates of statistical parameters, especially skewness and fourth-order cumulants which will be used in this work. The data analysis indicates that the statistics of unexpected waves can be based on Eq. (23), where the B_{rs} terms accounting for non-stationarity are neglected. Further, successive waves can be assumed as stochastically independent as the both sea states are broadbanded as indicated by their estimated covariance functions (see panel insets in Fig. 7).

As regard to WACSIS measurements, the left panel of Fig. 7 compares the theoretical nonlinear return period N_R (solid line) of unexpected wave crests α -times larger than the surrounding N_a waves, the respective linear predictions (dashed line) and the WACSIS empirical one-sided statistics for $\alpha = 1.5, 2$ (dashed line with \square). The right panel of the same figure shows similar comparisons for TERN. The observed occurrence rates are close to the theoretical predictions, indicating that the assumption of stochastic independence of waves holds approximately. It is noticed that nonlinearities tend to reduce the return period of unexpected waves and increase their mean crest amplitudes. In particular, in the left panel of Fig. 8 we compare our predicted nonlinear (solid line) and linear (dash line) mean crest heights $\bar{h}(\alpha, N_a)$ and $\bar{h}^{(L)}(\alpha, N_a)$ versus the WACSIS empirical one-sided statistics (\square) for $\alpha = 1.5$. Clearly, our linear predictions underestimate the observed crest amplitudes, as expected. Indeed, it is well established that nonlinearities must be accounted for to obtain reliable statistics of unexpected waves (Tayfun 1980; Forristall 2000; Tayfun and Fedele 2007; Fedele and Tayfun 2009; Gemmrich and Garrett 2011). Similar trend is also observed for the WACSIS rogue wave as evident from the center

panel of Fig. 9. Here, there are shown our nonlinear predicted mean crest height \bar{h}_{\max, N_R} , conditional mean h_{1/N_R} and mean unexpected crest height $\bar{h}_{\alpha=2, N_a}$ versus their linear counterparts. The right panel of the same figure depicts the nonlinear threshold h_{N_R} in comparison to its linear counterpart. The nonlinear and linear predictions for the Andrea rogue wave are also shown in Fig. 10.

We observe that the empirical statistics tend to deviate from the theoretical predictions for large values of α and N_a . In particular, for both TERN and WACSIS we could not produce statistically stable estimates of extreme values for $N_a > 10$ when $\alpha > 1.5$ due to the limited number of waves in the time series ($O(10^3)$ waves in comparison to the 10^6 waves of the simulated Gaussian seas). Nevertheless, the agreement between our present theory and observations is satisfactory and it also provides evidence that successive waves in the samples are approximately stochastically independent.

5. How rogue are unexpected waves?

WACSIS observations indicate that the actual largest crest h_{obs} is $1.62H_s$. Fig. 1 shows that the WACSIS rogue wave is also unexpected as it is $\alpha = 2$ -times larger than the surrounding $N_a \sim 50$ waves. According to our statistical model such unexpected wave would occur often and on average once every $N_R = 4 \cdot 10^4$ waves, as seen in the left panel of Fig. 9. Here, we report the theoretical predictions of the unconditional nonlinear return period N_R as a function of N_a using Eqs. (9) and (11)). Further, from the center panel of Fig. 9 it is seen that the associated average nonlinear unexpected crest height $\bar{h}_{(\alpha=2, N_a=50)}$ is about $1.35H_s$ and smaller than the conditional mean $h_{1/N_R} \sim 1.5H_s$, which is slightly larger than the mean maximum crest height $\bar{h}_{\max, N_R} = 1.48H_s$ of $N_R = 4 \cdot 10^4$ waves. Note that these average values underestimate the actual maximum crest amplitude $h_{obs} \sim 1.62H_s$ observed. In contrast, the right panel of Fig. 9 shows that h_{obs} is nearly the same as the threshold $h_{0.3,6} = 1.6H_s$ exceeded on average once every $N_h = 0.3 \cdot 10^6$ waves (see Eq. (17)).

We have seen that a correct statistical interpretation of the WACSIS rogue wave as an unexpected event requires considering the conditional return period $N_R(\xi)$ of an unexpected wave whose crest height is larger than ξH_s (see Eq. (24)). In particular, the left panel of Fig. 9 depicts plots of $N_R(\xi)$ as a function of N_a for increasing values of $\xi = 1, 1.2, 1.4, 1.55$ and 1.6 ($\alpha = 2$). For $\xi = 1.6H_s$, we find that an unexpected wave exceeding this threshold and standing above $N_a = 50$ waves would occur rarely and once every $N_R(\xi = 1.6) \sim 0.6 \cdot 10^6$, in contrast to the smaller unconditional value $N_R \sim 4 \cdot 10^4$.

In summary, the WACSIS wave crest as both unexpected and rogue, i.e. two-times larger than $N_a = 50$ surrounding waves and exceeding the $1.6H_s$, would occur once every $N_R = 0.6 \cdot 10^6$ waves on average. In contrast,

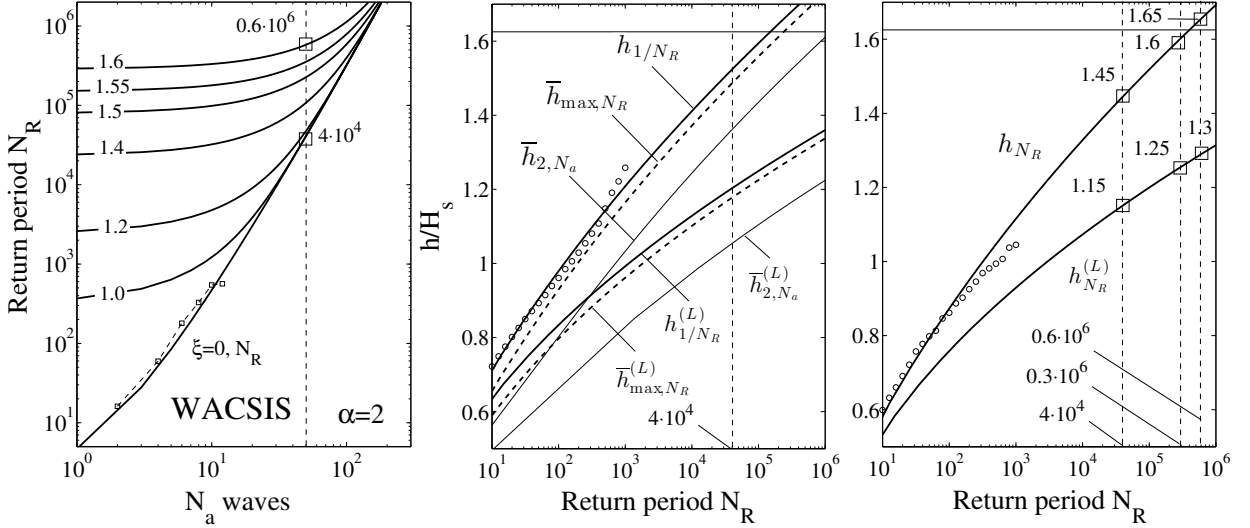


FIG. 9. WACSIS rogue wave: (Left panel) predicted nonlinear theoretical return periods $N_R(\xi)$, in number of waves, of unexpected crest heights greater than ξH_s and $\alpha = 2$ -times larger than the surrounding N_a waves for $\xi = 0, 1.0, 1.2, 1.4, 1.55$ and 1.6 (solid lines) and (square) empirical one-sided unexpected wave statistics. Dashed vertical line denotes return period values at $N_a = 50$. (Center panel) predicted nonlinear mean crest height \bar{h}_{\max, N_R} , conditional mean h_{1/N_R} and average unexpected crest height $\bar{h}_{\alpha=2, N_a}$ versus their linear counterparts as a function of number of waves N_R . Empirical conditional mean h_{1/N_R} is also shown (circles). (Right panel) predicted (solid line) and empirical (circles) nonlinear threshold h_{N_R} versus its linear counterpart as a function of N_R . Dashed vertical lines denote values at $N_R = 4 \cdot 10^4, 0.3 \cdot 10^6$ and $0.6 \cdot 10^6$. The horizontal line denotes the observed maximum crest height $1.62H_s$. Average wave parameters are taken from Tayfun (2006); Tayfun and Fedele (2007), in particular skewness $\lambda_3 \sim 0.23$ and excess kurtosis $\lambda_{40} \sim 0.11$.

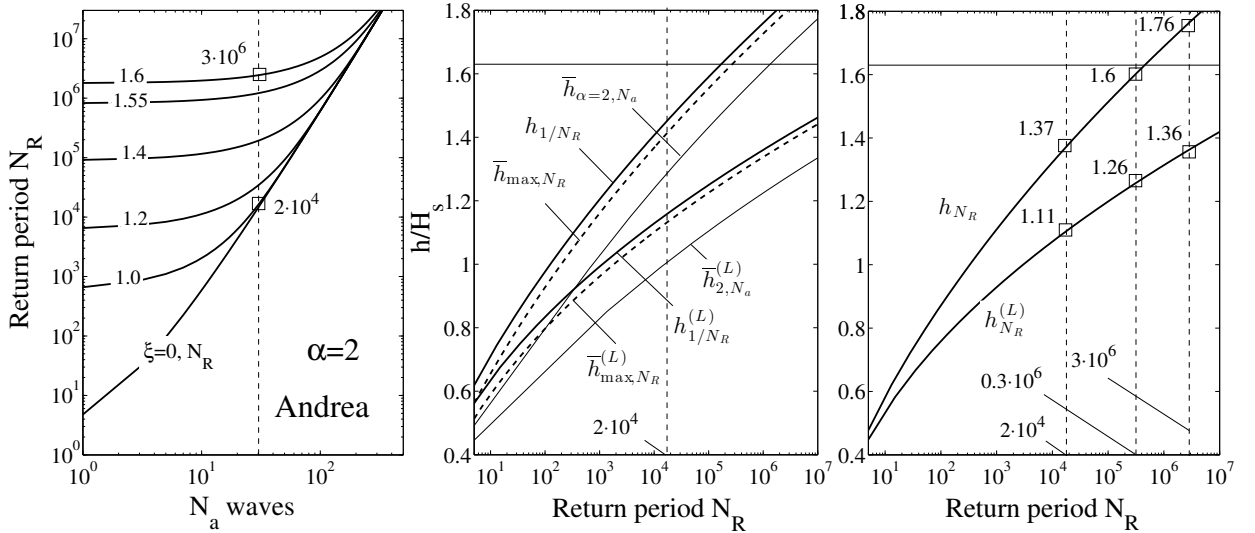


FIG. 10. Andrea rogue wave: (Left panel) predicted nonlinear theoretical return periods $N_R(\xi)$, in number of waves, of unexpected crest heights greater than ξH_s and $\alpha = 2$ -times larger than the surrounding N_a waves for $\xi = 0, 1.0, 1.2, 1.4, 1.55$ and 1.6 . Dashed vertical line denotes return period values at $N_a = 30$. (Center panel) predicted nonlinear mean crest height \bar{h}_{\max, N_R} , conditional mean h_{1/N_R} and average unexpected crest height $\bar{h}_{\alpha=2, N_a}$ versus their linear counterparts as a function of number of waves N_R . Empirical conditional mean h_{1/N_R} is also shown (circles). (Right panel) predicted (solid line) and empirical (circles) nonlinear threshold h_{N_R} versus its linear counterpart as a function of N_R . Dashed vertical lines denote values at $N_R = 2 \cdot 10^4, 0.3 \cdot 10^6$ and $3 \cdot 10^6$. The horizontal line denotes the observed maximum crest height $1.63H_s$. Wave parameters $H_s = 9.2$ m, $T_m = 13.2$ s, depth $d = 70$ m (Magnusson and Donelan 2013), skewness $\lambda_3 = 0.15$ and excess kurtosis $\lambda_{40} = 0.1$ (Dias et al. 2015).

the WACSIS wave as a rogue event has a crest height that is nearly the same as the threshold $h_{0.3 \cdot 10^6} = 1.6H_s$ ex-

ceeded on average once every $N_h = 0.3 \cdot 10^6$ waves. Thus, the WACSIS rogue wave has a slightly greater occurrence

frequency than being both rogue and unexpected since $N_h < N_R = 0.6 \cdot 10^6$. This implies that the threshold h_{N_R} exceeded by the $1/N_R$ fraction of the largest crests is larger than $1.6H_s$ and nearly the same as $1.65H_s$.

6. The Andrea rogue wave and its unexpectedness

As a specific application of the present theoretical framework, the unexpected wave statistics of the 2007 Andrea rogue wave event is examined. The actual largest crest height h_{obs} is $1.63H_s$ and nearly two-times larger than the surrounding $O(30)$ waves (see Fig. 12 in Magnusson and Donelan (2013)). For the hindcast Andrea sea state, the left panel of Fig. 10 shows the unconditional and conditional nonlinear return periods N_R and $N_R(\xi)$ as a function of N_a . In particular, according to our statistical model, the theoretical predictions indicate that a wave with a crest height at least twice that of any of the surrounding $N_a = 30$ waves occurs on average once every $N_R = 2 \cdot 10^4$ waves irrespective of its crest amplitude. In contrast, an unexpected wave whose crest height exceeds the threshold $1.6H_s$ occurs less often since our predicted conditional return period $N_R(\xi = 1.6) \sim 3 \cdot 10^6$ is greater than the unconditional counterpart $N_R = 2 \cdot 10^4$, as seen in the left panel of Fig. 10. Furthermore, the crest height $1.6H_s$ is nearly the same as the threshold $h_{0.3 \cdot 10^6}$ exceeded on average once every $N_h = 0.3 \cdot 10^6$ waves, as indicated in the right panel of the same figure. Thus, the Andrea wave has a greater occurrence rate than being both rogue and unexpected since $N_h < N_R = 3 \cdot 10^6$, and implying the larger threshold $h_{N_R} = 1.76H_s$.

7. Concluding remarks

We have presented a third-order nonlinear model for the statistics of unexpected waves. Gemmrich and Garrett (2008) define as unexpected a wave that is taller than a set of neighboring waves. The term "unexpected" refers to a wave that is not foreseen by a casual observer (Gemmrich and Garrett 2010). Clearly, unexpected waves are predictable in a statistical sense. Indeed, they can occur relatively often with a small or moderate crest height. However, unexpected waves that are rogue are rare. This difference in occurrence frequencies is quantified by introducing the conditional return period of an unexpected wave that exceeds a given threshold crest height. The associated unconditional return period is smaller than the conditional counterpart as it refers to the harmonic mean of the return periods of unexpected waves of any crest amplitude.

Furthermore, our analysis indicate that a wave that is both rogue and unexpected has a lower occurrence frequency than just being rogue. This is proven both analytically and verified by way of an analysis of the Andrea and WACSIS rogue wave events. Both waves appeared without warning and their crests were nearly 2-times larger

than the surrounding $O(10)$ wave crests, and thus unexpected. The two crest heights are nearly the same as the threshold $h_{0.3 \cdot 10^6} \sim 1.6H_s$ exceeded on average once every $0.3 \cdot 10^6$ waves. In contrast, the Andrea and WACSIS events would occur less often as being both unexpected and rogue, i.e. on average once every $3 \cdot 10^6$ and $0.6 \cdot 10^6$ respectively.

Finally, we point out that our statistical model for unexpected waves supports and goes beyond the analysis by Gemmrich and Garrett (2008) based on Monte Carlo simulations. In particular, our statistical approach can be used in operational wave forecast models to predict the unexpectedness of ocean waves.

8. Acknowledgments

FF is grateful to George Z. Forristall and M. Aziz Tayfun for sharing the wave measurements utilized in this study. FF thanks Michael Banner, George Forristall, Peter A. E. M. Janssen, Victor Shrira and M. Aziz Tayfun for discussions on nonlinear wave statistics. FF also thanks M. Aziz Tayfun for sharing his numerical solver for simulating second-order nonlinear waves. Further, FF thanks Michael Banner and M. Aziz Tayfun for revising an early draft of the manuscript as well as Guillermo Gallego for his support with L^AT_EX. FF acknowledges partial support from NSF grant CCF-1347191.

References

- Alkhalidi, M. A., and M. A. Tayfun, 2013: Generalized bocchetti distribution for nonlinear wave heights. *Ocean Engineering*, **74**, 101 – 106.
- Annenkov, S. Y., and V. I. Shrira, 2014: Evaluation of skewness and kurtosis of wind waves parameterized by jonswap spectra. *Journal of Physical Oceanography*, **44** (6), 1582–1594, doi:10.1175/JPO-D-13-0218.1, URL <http://dx.doi.org/10.1175/JPO-D-13-0218.1>.
- Bitner-Gregersen, E. M., L. Fernandez, J. M. Lefèvre, J. Monbaliu, and A. Toffoli, 2014: The north sea andrea storm and numerical simulations. *Natural Hazards and Earth System Science*, **14** (6), 1407–1415, doi:10.5194/nhess-14-1407-2014, URL <http://www.nat-hazards-earth-syst-sci.net/14/1407/2014/>.
- Bocchetti, P., 2000: *Wave Mechanics for Ocean Engineering*. Elsevier Sciences, Oxford, 496 pp.
- Borgman, L. E., 1970: Maximum wave height probabilities for a random number of random intensity storms. *ASCE Proceedings of 12th Conference on Coastal Engineering, Washington, D.C.*, <https://journals.tdl.org/icce/index.php/icce/article/view/2608>, Vol. 12, 53–64.
- Dias, F., J. Brennan, S. Ponce de Leon, C. Clancy, and J. Dudley, 2015: Local analysis of wave fields produced from hindcasted rogue wave sea states. *ASME 2015 34th International Conference on Ocean, Offshore and Arctic Engineering*, American Society of Mechanical Engineers, OMAE2015–41458.
- Dysthe, K. B., H. E. Krogstad, and P. Muller, 2008: Oceanic rogue waves. *Annual Review of Fluid Mechanics*, **40**, 287–310.

- Fedele, F., 2005: Successive wave crests in gaussian seas. *Probabilistic Engineering Mechanics*, **20** (4), 355 – 363, doi:http://dx.doi.org/10.1016/j.probengmech.2004.05.008, URL <http://www.sciencedirect.com/science/article/pii/S0266892005000317>.
- Fedele, F., 2012: Space-time extremes in short-crested storm seas. *Journal of Physical Oceanography*, **42** (9), 1601–1615, doi:10.1175/JPO-D-11-0179.1, URL <http://dx.doi.org/10.1175/JPO-D-11-0179.1>.
- Fedele, F., 2015a: On oceanic rogue waves. *arXiv preprint arXiv:1501.03370*.
- Fedele, F., 2015b: On the kurtosis of ocean waves in deep water. *Journal of Fluid Mechanics*, **782**, 25–36.
- Fedele, F., and M. A. Tayfun, 2009: On nonlinear wave groups and crest statistics. *J. Fluid Mech.*, **620**, 221–239.
- Forristall, G. Z., 2000: Wave crest distributions: Observations and second-order theory. *Journal of Physical Oceanography*, **30** (8), 1931–1943, doi:10.1175/1520-0485(2000)030<1931:WCDOAS>2.0.CO;2, URL [http://dx.doi.org/10.1175/1520-0485\(2000\)030<1931:WCDOAS>2.0.CO;2](http://dx.doi.org/10.1175/1520-0485(2000)030<1931:WCDOAS>2.0.CO;2).
- Forristall, G. Z., S. F. Barstow, H. E. Krogstad, M. Prevosto, P. H. Taylor, and P. S. Tromans, 2004: Wave crest sensor intercomparison study: An overview of wacsis. *Journal of Offshore Mechanics and Arctic Engineering*, **126** (1), 26–34, URL <http://dx.doi.org/10.1115/1.1641388>.
- Gemmrich, J., and C. Garrett, 2008: Unexpected waves. *Journal of Physical Oceanography*, **38** (10), 2330–2336.
- Gemmrich, J., and C. Garrett, 2010: Unexpected waves: Intermediate depth simulations and comparison with observations. *Ocean Engineering*, **37** (2–3), 262 – 267, doi:http://dx.doi.org/10.1016/j.oceaneng.2009.10.007, URL <http://www.sciencedirect.com/science/article/pii/S0029801809002467>.
- Gemmrich, J., and C. Garrett, 2011: Dynamical and statistical explanations of observed occurrence rates of rogue waves. *Natural Hazards and Earth System Science*, **11** (5), 1437–1446, doi:10.5194/nhess-11-1437-2011, URL <http://www.nat-hazards-earth-syst-sci.net/11/1437/2011/>.
- Haver, S., 2001: Evidences of the existence of freak waves. *Rogue Waves*, 129–140.
- Magnusson, K. A., and M. A. Donelan, 2013: The andrea wave characteristics of a measured north sea rogue wave. *Journal of Offshore Mechanics and Arctic Engineering*, **135** (3), 031 108–031 108, URL <http://dx.doi.org/10.1115/1.4023800>.
- Mori, N., and P. A. E. M. Janssen, 2006: On kurtosis and occurrence probability of freak waves. *Journal of Physical Oceanography*, **36** (7), 1471–1483, doi:10.1175/JPO2922.1, URL <http://dx.doi.org/10.1175/JPO2922.1>.
- Osborne, A. R., 1995: The numerical inverse scattering transform: nonlinear Fourier analysis and nonlinear filtering of oceanic surface waves. *Chaos Solitons Fractals*, **5** (12), 2623–2637.
- Tayfun, M. A., 1980: Narrow-band nonlinear sea waves. *Journal of Geophysical Research: Oceans*, **85** (C3), 1548–1552, doi:10.1029/JC085iC03p01548.
- Tayfun, M. A., 2006: Statistics of nonlinear wave crests and groups. *Ocean Engineering*, **33** (11–12), 1589 – 1622, doi:http://dx.doi.org/10.1016/j.oceaneng.2005.10.007.
- Tayfun, M. A., and F. Fedele, 2007: Wave-height distributions and nonlinear effects. *Ocean Engineering*, **34** (11–12), 1631 – 1649.
- Tayfun, M. A., and J. Lo, 1990: Nonlinear effects on wave envelope and phase. *J. Waterway, Port, Coastal and Ocean Eng.*, **116**, 79–100.
- Watson, G. S., 1954: Extreme values in samples from m -dependent stationary stochastic processes. *Ann. Math. Statist.*, **25** (4), 798–800, doi:10.1214/aoms/1177728670, URL <http://dx.doi.org/10.1214/aoms/1177728670>.